# Event-Triggered Fault Reconstruction Using Sliding Mode Observer ⋆

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**Abstract:** In this paper, we study event-triggered fault reconstruction using sliding mode observer (SMO) for a linear time-invariant system. Here, the SMO is designed with an event condition in which output measurements are transmitted over the network. The observer provides the state estimates using the sampled output values in the presence of fault signals. It is shown that with a suitable (switching) gain and the triggering parameter, the observer states converge to the true states (in the practical sense), and thus, the fault signals can be reconstructed with an accuracy dependent on the steady-state error bound. Finally, simulation results are shown to illustrate fault detection employing the proposed approach.

*Keywords:* Event-based feedback, fault reconstruction, sliding mode observer.

## 1. INTRODUCTION

Fault detection and isolation (FDI) are essential in resolving problems related to economic loss and safety-relevant issues. Generally, unpredictable changes in plants, such as component malfunctioning or sudden changes in operating conditions, known as a fault, may cause a complete system failure. Thus, early detection and isolation of faults are always preferred to avoid any loss or damage to the system (Frank (1990)). Model-based FDI techniques are very common in literature because the residual signal is generally used as an indicator of fault (Edwards et al. (2000)). Different approaches have been employed in the generation of residuals, such as the parity space approach (Patton and Chen (1991)), dedicated observer approach (Frank (1990)), fault detection filter (Edwards et al. (2000)), etc. The fundamental concept in all these approaches is the use of the state estimation technique to detect the fault. The application of Luenberger observer is found in Patton and Chen (1991) in the generation of residuals. Hermans and Zarrop (1996), Sreedhar et al. (1993) incorporated the sliding mode observer (SMO) in the FDI problem. Indeed, Sreedhar et al. (1993) assumed the availability of all the state variables for the detection of faults, whereas Hermans and Zarrop (1996) considered a general multi-input multi-output system for FDI. Hermans and Zarrop (1996) showed that sliding motion ceases if the external perturbation goes beyond some acceptable limit which indicates the occurrence of faults.

In some applications, the information about the size and severity of faults are important (Alwi et al. (2011), Edwards and Spurgeon (2000)). So, along with the detection and isolation, the reconstruction of faults is also necessary for the fault diagnosis scheme. The use of SMO can be noticed in Edwards et al. (2000), where the authors utilized the discontinuous switching function for reconstructing the faults under some assumptions. Here, the authors also considered a general multi-input multi-output system for FDI scheme and shown that along with the detection of fault, reconstruction of fault is possible without the cease of sliding motion. A similar approach for fault estimation is reported in Edwards and Spurgeon (2000) for the ship benchmark problem. In addition, Alwi et al. (2011) showed that using Utkin observer, one can also estimate a certain type of disturbances.

The extension of the fault reconstruction using SMO in the sampled-data framework is not straightforward. This is due to the availability of system information to the observer at discrete time instants only as opposed to the analog case. In the last decade, few researchers have shown interest in addressing this issue. The estimation of some external disturbances for a linear time-invariant (LTI) system in the sampled-data framework is seen in Nguyen et al. (2021). Here, the authors employed SMO to reconstruct the disturbance in a discrete-time setup. Han et al. (2011) also considered the (conventional) discrete data transmission for fault estimation but utilized the time delay approach to solve the problem. It should be noted that due to the regular sampling, the approximation of fault estimation depends on the sampling period; the smaller the sampling period better the accuracy, and vice versa. As the high sampling frequency increases the burden on the communication network, it becomes a challenge to implement the traditional techniques in resource-constrained systems. It is seen in Tabuada (2007); Lunze and Lehmann (2010); Tallapragada and Chopra (2013); Behera et al. (2021); Behera and Bandyopadhyay (2014); Bandyopadhyay and Behera (2018); Kumari et al. (2020); Cucuzzella et al. (2020) that the communication channel can be utilized optimally by using an event-based data transmission strategy instead of the conventional one.

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The advantage of event-triggered data transmission is also seen in the estimation problem, e.g., Luenberger observer (Zhang and Feng (2014)), SMO (Amara et al. (2020)), etc. This motivates us to employ an event-triggered data transmission strategy in fault reconstruction. In this paper, we consider a multi-input multi-output LTI system that is under the influence of actuator fault. Here, we employ SMO to estimate the fault by considering that the system information is available over the communication network at discrete time instants only. The instant of data transmission is decided by some specific condition, called an event. Indeed, we show that the state estimation error can be bounded within an arbitrary bound so that the fault signal can be reconstructed with an accuracy dependent on the estimation error.

The layout of the paper is organized as follows: The problem is formulated in Section 2. In Section 3, considering event-triggered data transmission, the SMO is designed, which is followed by the main results of the paper in Section 4. Here, we establish that the state estimation errors can be bounded with any region, and then, the fault can be reconstructed depending on the accuracy of this error. A practical example is taken to illustrate the proposed strategy in Section 5. Finally, we concluded the paper with some remarks in Section 6.

*Notation:* The set of real numbers and integers are represented by $\mathbb{R}$ and $\mathbb{Z}$, respectively. Given a square matrix $W \in \mathbb{R}^{n\times n}$, the minimum and maximum eigenvalues are denoted by $\lambda_m(W)$ and $\lambda_M(W)$, respectively. The 1-norm and (Euclidean) 2-norm are given by $\|x\|_1$ and $\|x\|_2$ (or $\|x\|$) for any vector $x \in \mathbb{R}^n$. Given a matrix $W \in \mathbb{R}^{m\times n}$, $\|W\|$ is the matrix norm induced by the vector 2-norm. The rank of $P \in \mathbb{R}^{m\times n}$ is denoted by $\operatorname{rank}(P)$. Similarly, $P^\top$ is the transpose of the matrix $P$ and $P > 0$ is the positive definiteness of $P$. A diagonal matrix is written as $\operatorname{diag}\{a_1, a_2, \dots, a_n\}$, where $a_i$ is the $i$th diagonal element. The absolute value of a scalar $a$ is denoted by $|a|$, whereas $|\Gamma|$ denotes the cardinality when $\Gamma$ is a set.

## 2. PROBLEM STATEMENT

Consider an LTI system as

$$\dot{x} = Ax + Bu + Qf_a \tag{1a}$$
$$y = Cx \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output vector and the function $f_a \in \mathbb{R}^q$ is the unknown actuator fault signal. The matrices $A$, $B$, $C$, and $Q$ are of appropriate dimensions and are assumed to be known to the designer. Here, the vector $f_a$ characterizes the presence of fault signals in the actuators. The nonzero components of $f_a$ denote that the actuators are corrupted by the fault signal. Otherwise, all the actuators are operating under normal conditions when $f_a = 0$. Moreover, it is considered that $p > q$ throughout the paper.

**Assumption 1.** The invariant zeros of $(A, Q, C)$ are located in the open left half of the complex plane.

**Assumption 2.** $C$ is a full rank matrix and $\operatorname{rank}(CQ) = q$.

**Assumption 3.** $\sup_{t\geq 0} \|f_a(t)\| \leq f_0$ for a known constant $f_0 > 0$.

Assumptions 1 and 2 above are required to construct an SMO that is robust to the actuator fault. In fact, the relative degree requirement stems from the sliding mode constraint of the observer dynamics. Assumption 3 is justifiable as the magnitude of the extreme fault signals is generally available from the experimental knowledge. In addition to these assumptions, we consider that the plant is situated at a remote location, so the information exchange takes place only through communication channels. This has led to another challenge to the traditional fault detection techniques as the data are available intermittently. We employ here an event-triggering data transmission strategy to reduce the communication burden among the various components of the system.

The event detector generates the time instants at which the plant information is transmitted. We denote by $\{t_k\}_{k=0}^{\infty}, k \in \mathbb{Z}_{\geq 0}$ the sequence of these triggering instants. The sampled output, $y(t_k)$, is transmitted over the network to an observer that detects whether the actuators are affected by fault signals. Then, this observer reconstructs the fault with an accuracy subject to the event parameter. The rest of the paper is devoted to the synthesis of fault reconstruction under event-triggering feedback.

## 3. EVENT BASED SMO

This section proposes an event-triggered SMO for constructing the fault signals in the plant. We transform the system (1) into a suitable canonical structure to facilitate our analysis.

Let $N_c \in \mathbb{R}^{n\times(n-p)}$ be the matrix whose columns span the null space of $C$, i.e., $CN_c = 0$. Taking the (nonsingular) transformation

$$T = \begin{bmatrix} N_c^\top \\ C \end{bmatrix},$$

the plant (1) can be transformed in the new coordinate, $\mathbf{x} = [\,\mathbf{x}_1^\top \;\; y^\top\,]^\top = Tx$, as

$$\dot{\mathbf{x}}_1 = \overline{A}_{11}\mathbf{x}_1 + \overline{A}_{12}y + \overline{B}_1 u + \overline{Q}_1 f_a \tag{2a}$$
$$\dot{y} = \overline{A}_{21}\mathbf{x}_1 + \overline{A}_{22}y + \overline{B}_2 u + \overline{Q}_2 f_a \tag{2b}$$
$$y = \overline{C}\mathbf{x} \tag{2c}$$

where $\mathbf{x}_1 \in \mathbb{R}^{n-p}$ and $y \in \mathbb{R}^p$ are the first $n-p$ and last $p$ components of $\mathbf{x}$, respectively. The various matrices in (2) are given by

$$\overline{A} = TAT^{-1} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix},\ \overline{B} = TB = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$$

$$\overline{Q} = TQ = \begin{bmatrix} \overline{Q}_1 \\ \overline{Q}_2 \end{bmatrix} \text{ and } \overline{C} = CT^{-1} = [\,0 \;\; I_p\,]$$

where [1] $T^{-1} = [\,N_c(N_c^\top N_c)^{-1} \;\; C^\top(CC^\top)^{-1}\,]$. One of the immediate observations is that the last $p$ state variables are the outputs of the system (2). We apply another transformation to the system (2). Since $\overline{Q}_2$ has full rank, we can write $\overline{Q}_2 = [\,\overline{Q}_{21}^\top \;\; \overline{Q}_{22}^\top\,]^\top$ such that $\overline{Q}_{22} \in \mathbb{R}^{q\times q}$ is invertible. Then, it can be easily verified for

$$W_2 = \begin{bmatrix} I_{p-q} & -\overline{Q}_{21}\overline{Q}_{22}^{-1} \\ 0 & I_q \end{bmatrix}$$

[1] It can be verified that $TT^{-1} = T^{-1}T = I_n$.

that

$$W_2\overline{Q}_2 = \begin{bmatrix} 0 \\ \overline{Q}_{22} \end{bmatrix}.$$

Now, by constructing an invertible matrix

$$W = \begin{bmatrix} I_{n-p} & -\overline{Q}_1(\overline{Q}_2^\top \overline{Q}_2)^{-1}\overline{Q}_2^\top \\ 0 & W_2 \end{bmatrix},$$

we again observe that

$$W\overline{Q} = \begin{bmatrix} 0 \\ 0 \\ \overline{Q}_{22} \end{bmatrix}.$$

Taking the transformation, $\mathbf{x} \mapsto W\mathbf{x} = [\,\overline{\mathbf{x}}_1^\top \; \overline{y}_1^\top \; y_2^\top\,]^\top$, the system (2) can be transformed into the form

$$\begin{aligned} \dot{\overline{\mathbf{x}}}_1 &= \overline{\mathbf{A}}_{11}\overline{\mathbf{x}}_1 + \overline{\mathbf{A}}_{1211}\overline{y}_1 + \overline{\mathbf{A}}_{1212}y_2 + \overline{\mathbf{B}}_1 u \\ \dot{\overline{y}}_1 &= \overline{\mathbf{A}}_{2111}\overline{\mathbf{x}}_1 + \overline{\mathbf{A}}_{2211}\overline{y}_1 + \overline{\mathbf{A}}_{2212}y_2 + \overline{\mathbf{B}}_{21} u \\ \dot{y}_2 &= \overline{\mathbf{A}}_{2112}\overline{\mathbf{x}}_1 + \overline{\mathbf{A}}_{2221}\overline{y}_1 + \overline{\mathbf{A}}_{2222}y_2 + \overline{\mathbf{B}}_{22} u \\ &\quad + \overline{\mathbf{Q}}_{22} f_a. \end{aligned} \tag{3}$$

Note that here $\overline{\mathbf{Q}}_{22} = \overline{Q}_{22}$. One can see that the unknown fault only appears in the last $q$ output dynamics of the transformed system. So, the remaining $p-q$ outputs can be used to design the observer gain for the robust fault reconstruction. This is one of the essential features of the SMO. Again, to facilitate the observer design, we further represent the system in a new coordinate.

Assumption 1 guarantees that $(\overline{\mathbf{A}}_{11}, \overline{\mathbf{A}}_{2111})$ is a detectable pair. So, there exists an $L_1 \in \mathbb{R}^{(n-p)\times(p-q)}$ such that $\overline{\mathbf{A}}_{11} + L_1\overline{\mathbf{A}}_{2111}$ is Hurwitz. Applying the following coordinate change

$$W\mathbf{x} \mapsto z = \begin{bmatrix} I_{n-p} & L_1 & 0 \\ 0 & I_{p-q} & 0 \\ 0 & 0 & I_q \end{bmatrix} W\mathbf{x},$$

we can express the system (3) as

$$\dot{z}_1 = \widetilde{A}_{11}z_1 + \widetilde{A}_{1211}z_{21} + \widetilde{A}_{1212}z_{22} + \widetilde{B}_1 u \tag{4a}$$

$$\dot{z}_{21} = \widetilde{A}_{2111}z_1 + \widetilde{A}_{2211}z_{21} + \widetilde{A}_{2212}z_{22} + \widetilde{B}_{21} u \tag{4b}$$

$$\dot{z}_{22} = \widetilde{A}_{2112}z_1 + \widetilde{A}_{2221}z_{21} + \widetilde{A}_{2222}z_{22} + \widetilde{B}_{22} u + \widetilde{Q}_{22} f_a \tag{4c}$$

where $z = [\,z_1^\top \; z_{21}^\top \; z_{22}^\top\,]^\top$, $\widetilde{A}_{11} = \widetilde{A}_{11} + L_1\widetilde{A}_{2111}$ and $\widetilde{Q}_{22} = \overline{Q}_{22}$. To avoid the notational complexities, we rewrite (4) as

$$\dot{z}_1 = \widetilde{A}_{11}z_1 + \widetilde{A}_{12}z_2 + \widetilde{B}_1 u \tag{5a}$$

$$\dot{z}_2 = \widetilde{A}_{21}z_1 + \widetilde{A}_{22}z_2 + \widetilde{B}_2 u + \widetilde{Q}_2 f_a \tag{5b}$$

where $z_2 = [\,z_{21}^\top \; z_{22}^\top\,]^\top$ and $\widetilde{Q}_2 = [\,0^\top \; \widetilde{Q}_{22}^\top\,]^\top$. Now onwards, we use the system (4) and (5) interchangeably in the analysis.

Let $\xi(t) = z_2(t) - z_2(t_k)$ be the output sampling error where $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}_{\geq 0}$. Then, we propose the following event-triggering mechanism

$$\begin{aligned} t_0 &= 0, \\ t_{k+1} &= \inf\{t > t_k : \|\xi(t)\| \geq \sigma\alpha\}, \end{aligned} \tag{6}$$

where $\alpha$ is the design parameter which is a positive scalar quantity, and $\sigma \in (0,1)$ is an additional factor used to tighten the condition further. When the event condition is satisfied, the plant outputs are transmitted over the communication network to the SMO.

Choose any stable matrix $\widetilde{A}_{22}^s$. Obtain the matrix $P_2 = P_2^\top > 0$ satisfying

$$(\widetilde{A}_{22}^s)^\top P_2 + P_2\widetilde{A}_{22}^s = -N_2 \tag{7}$$

for any $N_2 = N_2^\top > 0$. With these parameters, we propose the following observer

$$\begin{aligned} \dot{\hat{z}}_1(t) &= \widetilde{A}_{11}\hat{z}_1(t) + \widetilde{A}_{12}\hat{z}_2(t) + \widetilde{B}_1 u(t) \\ &\quad + G_1(\hat{z}_2(t) - z_2(t_k)) \end{aligned} \tag{8a}$$

$$\begin{aligned} \dot{\hat{z}}_2(t) &= \widetilde{A}_{21}\hat{z}_1(t) + \widetilde{A}_{22}\hat{z}_2(t) + \widetilde{B}_2 u(t) + \nu(t) \\ &\quad + G_2(\hat{z}_2(t) - z_2(t_k)) \end{aligned} \tag{8b}$$

where $G_1 = -\widetilde{A}_{12}$, $G_2 = \widetilde{A}_{22}^s - \widetilde{A}_{22}$, and $t \in [t_k, t_{k+1})$ for any $k \in \mathbb{Z}_{\geq 0}$. Here, $\hat{z} = [\,\hat{z}_1^\top \; \hat{z}_2^\top\,]^\top$ is the estimate of the state $z$, and the discontinuous switching function $\nu(t)$ is given by

$$\nu(t) = -MP_2^{-1}\operatorname{sign}(z_2(t) - z_2(t_k)) \tag{9}$$

where $M > 0$ is the switching gain to be designed later and $\operatorname{sign}(\cdot)$ is taken componentwise. This observer with the event-triggering mechanism (6) is to be used to reconstruct the actuator fault signal.

## 4. EVENT BASED FAULT RECONSTRUCTION

This section presents the event-triggered fault reconstruction using the observer (8). In the analog feedback case, SMO can construct the fault exactly (Edwards et al. (2000)) (when there is no disturbance). When the event-triggered feedback is considered, the accuracy of the estimated fault signal is dependent on the triggering parameter (as we see later). However, one can still reconstruct the fault in the practical sense, which is comparable to the analog case.

Define the estimation errors as $e_1 = \hat{z}_1 - z_1$ and $e_2 = \hat{z}_2 - z_2$. Then, we can write the estimation error dynamics using (5) and (8) as

$$\dot{e}_1(t) = \widetilde{A}_{11}e_1(t) - \widetilde{A}_{12}\xi(t) \tag{10a}$$

$$\begin{aligned} \dot{e}_2(t) &= \widetilde{A}_{21}e_1(t) + \widetilde{A}_{22}^s e_2(t) + (\widetilde{A}_{22}^s - \widetilde{A}_{22})\xi(t) \\ &\quad - \widetilde{Q}_2 f_a(t) + \nu(t). \end{aligned} \tag{10b}$$

It can be observed that the fault signal only affects the output error dynamics, whereas the sampling error appears in the whole dynamics. The effects of these signals can be minimized by appropriately designing the switching observer gain and the triggering parameter. In fact, we show that the fault signal can be recovered with any given accuracy. When there is no sampling error in (10), the fault signal can be recovered accurately. As shown in Alwi et al. (2011); Edwards et al. (2000), the sliding motion takes place on $\{e = [\,e_1^\top \; e_2^\top\,]^\top \in \mathbb{R}^n : e_2 = 0\}$ for some sufficiently large $M$. Moreover, the trajectory $e_1(t)$ converges to zero asymptotically. By recalling that (the equivalent value of) $\dot{e}_2 = 0$, it is possible to recover the fault signal exactly using the relation

$$\nu_2 = \widetilde{Q}_{22} f_a \iff f_a = \widetilde{Q}_{22}^{-1}\nu_2$$

where $\nu_2 \in \mathbb{R}^q$ is the last $q$ components of $\nu$ ($\nu_1 \in \mathbb{R}^{p-q}$ is the first $p-q$ components of $\nu$). Similar arguments are also used in the event-triggering framework for reconstructing the fault signal.

By noting that $\widetilde{A}_{11}$ is Hurwitz, we obtain the matrix $P_1 = P_1^\top > 0$ as a solution to the Lyapunov equation

$$\widetilde{A}_{11}^\top P_1 + P_1\widetilde{A}_{11} = -N_1 \tag{11}$$

for any $N_1 = N_1^\top > 0$. Construct the following sets: $\Pi_1 = \{z_1 \in \mathbb{R}^{n-p} : z_1^\top P_1 z_1 \le c_1\}$ and $\Pi_2 = \{z_2 \in \mathbb{R}^p : z_2^\top P_2 e_2 \le c_2\}$ for some positive constants $c_1$ and $c_2$. Finally, define

$$\Pi = \{z \in \mathbb{R}^n : z_1 \in \Pi_1 \text{ and } z_2 \in \Pi_2\}.$$

The above sets are required in the analysis of an event-triggered observer. As the triggering mechanism is process dependent, the boundedness of the output trajectory is required to ensure the absence of Zeno behavior. Then, the following result is immediate.

**Proposition 1.** Consider the system (5) and the triggering mechanism (6). Suppose that $z(t) \in \Pi$ for all $t \ge 0$. Then, for any $\alpha > 0$, there exists a $\tau > 0$ such that $t_{k+1} - t_k \ge \tau$ for all $k \in \mathbb{Z}_{\ge 0}$.

**Proof.** First, let us write (5) as

$$\begin{aligned}\dot{z} &= \widetilde{A}z + \widetilde{B}u + \widetilde{Q}f_a\\ y &= \widetilde{C}z\end{aligned}$$

where the matrices above carry the usual meaning and are given by

$$\widetilde{A} = \begin{bmatrix}\widetilde{A}_{11} & \widetilde{A}_{12}\\ \widetilde{A}_{21} & \widetilde{A}_{22}\end{bmatrix},\quad \widetilde{B} = \begin{bmatrix}\widetilde{B}_1\\ \widetilde{B}_2\end{bmatrix},\quad \widetilde{Q} = \begin{bmatrix}0\\ \widetilde{Q}_2\end{bmatrix}$$

and $\widetilde{C} = \overline{C}W^{-1}$. Note that since the plant state is bounded, the output trajectory is also bounded. Then, for all $t \in [t_k, t_{k+1}) \setminus \{t \in [t_k, t_{k+1}) : \|\xi(t)\| = 0\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\| \le \left\|\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)\right\| = \left\|\widetilde{C}(\widetilde{A}z(t) + \widetilde{B}u(t) + \widetilde{Q}f_a(t))\right\|.$$

Generally, the control input is bounded due to physical limitations on the actuator. Then, Assumption 2 guarantees that there exists a constant $\beta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\| \le \beta, \quad \forall\, t \ge 0.$$

Solving this differential inequality using Comparison Lemma (Khalil (2002)) with $\|\xi(t_k)\| = 0$, we obtain that $\|\xi(t)\| \le \beta(t - t_k)$ for all $t \in [t_k, t_{k+1})$. At triggering instant, this is reduced to $\sigma\alpha \le \beta(t_{k+1} - t_k)$. By noting $k$ is arbitrary, we conclude that $t_{k+1} - t_k \ge \tau := (\sigma\alpha/\beta)$ for all $k \in \mathbb{Z}_{\ge 0}$, which ends the proof. $\square$

The above result ensures no accumulation of triggering instants; thus, our triggering mechanism (6) is a stabilizing condition. The only criterion required to ensure a positive lower bound between any two consecutive triggering instants is the boundedness of the state trajectory, which is achieved by designing a stabilizing feedback law. Under this consideration now, we present our main result below.

**Theorem 1.** Consider the dynamics (10) and the event-triggering mechanism (6). Suppose that $z(t) \in \Pi$ for all $t \ge 0$, and moreover $\hat{z}(0) \in \Pi$. Then, for every $\varepsilon > 0$, there exist $M > 0$, $\alpha > 0$, $\varepsilon_a > 0$ and $\tau_a > 0$ such that

- $\|e(t)\| < \varepsilon$ for all $t \ge \tau_a$,
- $\|\widetilde{Q}_{22}^{-1}\nu_2(t) - f_a(t)\| < \varepsilon_a$ for all $t \ge \tau_a$.

**Proof.** Let us define the Lyapunov function $V_1(e_1) = e_1^\top P_1 e_1$. Differentiating $V_1$ with respect to time along the solutions of (10a), one gets

$$\begin{aligned}\dot{V}_1(e_1) &= e_1^\top(\widetilde{A}_{11}^\top P_1 + P_1\widetilde{A}_{11})e_1 - 2e_1^\top P_1\widetilde{A}_{12}\xi\\ &= -e_1^\top N_1 e_1 - 2e_1^\top P_1\widetilde{A}_{12}\xi\\ &\le -\lambda_m(N_1)\|e_1\|^2 + 2\|e_1\|\|P_1\widetilde{A}_{12}\|\|\xi\|\\ &\le -\frac{\lambda_m(N_1)}{2}\|e_1\|^2 + \frac{2\|P_1\widetilde{A}_{12}\|^2}{\lambda_m(N_1)}\|\xi\|^2\\ &\le -\frac{\lambda_m(N_1)}{2\lambda_M(P_1)}V_1(e_1) + \frac{2\|P_1\widetilde{A}_{12}\|^2}{\lambda_m(N_1)}\|\xi\|^2.\end{aligned} \tag{12}$$

Note that we applied Young's inequality (Khalil (2002)) to the second term above and used the inequality $V_1(e_1) \le \lambda_M(P_1)\|e_1\|^2$. From the triggering condition, we know that $\|\xi(t)\| < \alpha$ for all $t \ge 0$. So,

$$\begin{aligned}\dot{V}_1(e_1) &< -\frac{\lambda_m(N_1)}{2\lambda_M(P_1)}V_1(e_1) + \frac{2\|P_1\widetilde{A}_{12}\|^2}{\lambda_m(N_1)}\alpha^2\\ &\le -\frac{\lambda_m(N_1)}{4\lambda_M(P_1)}V_1(e_1)\end{aligned}$$

whenever $V_1(e_1(t)) \ge \theta\alpha^2$ where $\theta = [8\lambda_M(P_1)\|P_1\widetilde{A}_{12}\|^2 / \lambda_m^2(N_1)]$. By taking $c_1 > \lambda_m(P_1)\theta\alpha^2/(4\lambda_M(P_1))$ and

$$\alpha \le \sqrt{\frac{\varepsilon^2\lambda_m(P_1)}{4\theta}}, \tag{13}$$

it can be concluded that the trajectories of (10a) starting within the set $\{e_1 \in \mathbb{R}^{n-p} : V_1(e_1) \le 4c_1\lambda_M(P_1)/\lambda_m(P_1)\}$ are located in the set $\{e_1 \in \mathbb{R}^{n-p} : V_1(e_1) < \lambda_m(P_1)\varepsilon^2/4\}$ in a finite time $\tau_{a_1} \ge 0$. Since $\lambda_m(P_1)\|e_1\|^2 \le V_1(e_1)$, it holds that $\|e_1(t)\| < \varepsilon/2$ for all $t \ge \tau_{a_1}$.

Now, our aim is to analyze the dynamics of $e_2(t)$, which is required for estimating the actuator fault. We take $V_2(e_2) = e_2^\top P_2 e_2$ as the Lyapunov function for the dynamics (10b). The time derivative of $V_2$ shows that

$$\begin{aligned}\dot{V}_2(e_2) &= e_2^\top((\widetilde{A}_{22}^s)^\top P_2 + P_2\widetilde{A}_{22}^s)e_2 + 2e_2^\top P_2\widetilde{A}_{21}e_1\\ &\quad - 2e_2^\top P_2\widetilde{Q}_2 f_a + 2e_2^\top P_2(A_{22}^s - \widetilde{A}_{22})\xi + 2e_2^\top P_2\nu\\ &= -e_2^\top N_2 e_2 + 2e_2^\top P_2\widetilde{A}_{21}e_1 - 2e_2^\top P_2\widetilde{Q}_2 f_a\\ &\quad + 2e_2^\top P_2(A_{22}^s - \widetilde{A}_{22})\xi + 2e_2^\top P_2\nu\\ &< -\lambda_m(N_2)\|e_2\|^2 + 2\|P_2\widetilde{A}_{21}e_1\|\|e_2\| + 2\nu^\top P_2 e_2\\ &\quad + 2\alpha\|P_2(\widetilde{A}_{22}^s - \widetilde{A}_{22})\|\|e_2\| + 2f_0\|P_2\widetilde{Q}_2\|\|e_2\|.\end{aligned} \tag{14}$$

Consider the set $\Pi_\varepsilon = \{e_2 = [\,e_{2_1} \cdots e_{2_p}\,]^\top \in \mathbb{R}^p : V_2(e_2) \le \lambda_m(P_2)\varepsilon^2/4, |e_{2_i}| \le \rho_i, i = 1, 2, \ldots, p\}$ for largest scalars $\rho_i > 0$ such that $\{e_2 \in \mathbb{R}^p : |e_{2_i}| \le \rho_i, i = 1, 2, \ldots, p\} \subseteq \Pi_\varepsilon$. If $e_2$ is outside the set $\Pi_\varepsilon$, then $|e_{2_i}| > \rho_i$ and so, $\nu^\top P_2 e_2 = -M\|e_2\|_1$. Therefore, the above inequality (14) is reduced to

$$\begin{aligned}\dot{V}_2(e_2) &< -\frac{\lambda_m(N_2)}{\lambda_M(P_2)}V_2(e_2) + 2\alpha\|P_2(\widetilde{A}_{22}^s - \widetilde{A}_{22})\|\|e_2\|\\ &\quad + 2\|P_2\widetilde{A}_{21}e_1\|\|e_2\| + 2f_0\|P_2\widetilde{Q}_2\|\|e_2\| - 2M\|e_2\|_1\\ &\le -\frac{\lambda_m(N_2)}{\lambda_M(P_2)}V_2(e_2) - 2(M - \|P_2\widetilde{A}_{21}e_1\|\\ &\quad - f_0\|P_2\widetilde{Q}_2\| - \alpha\|P_2(\widetilde{A}_{22}^s - \widetilde{A}_{22})\|)\|e_2\|.\end{aligned} \tag{15}$$

As we know the fact that $e_1(t) \in \Pi_1$ for all $t \geq 0$, choosing the switching gain

$$M \geq \|P_2\widetilde{A}_{21}e_1\| + \alpha\|P_2(\widetilde{A}^s_{22} - \widetilde{A}_{22})\| + f_0\|P_2\widetilde{Q}_2\| + \eta$$

for some $\eta > 0$ and using it in (15) leads to

$$\dot{V}_2(e_2) < -2\eta\|e_2\| \leq -2\eta\sqrt{V_2(e_2)}/\sqrt{\lambda_M(P_2)}.$$

This shows that $\Pi_\varepsilon$ is attractive when $e_2$ is outside of it. Now, let $\Gamma \subset \{1, 2, \ldots, p\}$ be the set of indices such that $|e_{2_i}| \leq \rho_i$ for all $i \in \Gamma$. Denote $e_{2_{\overline{\Gamma}}} \in \mathbb{R}^{|\overline{\Gamma}|}$ as the vector obtained by retaining those components of $e_2$ which lie in $\overline{\Gamma} = \{1, 2, \ldots, p\} \setminus \Gamma$. Then, the dynamics of $e_{2_{\overline{\Gamma}}}$ can be given by

$$\dot{e}_{2_{\overline{\Gamma}}} = \widetilde{A}_{21_{\overline{\Gamma}}}e_1 + \widetilde{A}^s_{22_{\overline{\Gamma}}}e_{2_{\overline{\Gamma}}} + (\widetilde{A}^s_{22_{\overline{\Gamma}}} - \widetilde{A}_{22_{\overline{\Gamma}}})\xi_{\overline{\Gamma}} - \widetilde{Q}_{2_{\overline{\Gamma}}}f_a + \nu_{\overline{\Gamma}}.$$

Clearly, $\widetilde{A}^s_{22_{\overline{\Gamma}}}$ is stable matrix. So, there exists $P_{2_{\overline{\Gamma}}} = P^\top_{2_{\overline{\Gamma}}} > 0$ such that $(\widetilde{A}^s_{22_{\overline{\Gamma}}})^\top P_{2_{\overline{\Gamma}}} + P_{2_{\overline{\Gamma}}}\widetilde{A}^s_{22_{\overline{\Gamma}}} = -Q_{2_{\overline{\Gamma}}}$ for some $Q_{2_{\overline{\Gamma}}} = Q^\top_{2_{\overline{\Gamma}}} > 0$. It can be shown for $V_2(e_{2_{\overline{\Gamma}}}) = (e^\top_{2_{\overline{\Gamma}}})P_{2_{\overline{\Gamma}}}e_{2_{\overline{\Gamma}}}$ that

$$\dot{V}_2(e_{2_{\overline{\Gamma}}}) < -2\eta\sqrt{V_2(e_{2_{\overline{\Gamma}}})}/\sqrt{\lambda_M(P_{2_{\overline{\Gamma}}})}$$

whenever $|e_{2_i}| > \rho_i$ for all $i \in \overline{\Gamma}$. Take $c_2 > \lambda_m(P_2)\varepsilon^2/(16\lambda_M(P_2))$ and $\alpha < \varepsilon/2$. Then, the trajectory $e_2$ starting from the set $\{e_2 \in \mathbb{R}^p : V_2(e_2) \leq 4c_2\lambda_M(P_2)/\lambda_m(P_2)\}$ will converge to the set $\Pi_\varepsilon$ in a finite time $\tau_{a_2} \geq 0$. Since $\lambda_m(P_2)\|e_2\|^2 \leq V_2(e_2)$ and $\|\xi(t)\| < \alpha$ in any triggering interval $t_k \leq t < t_{k+1}$, it holds immediately that $\|e_2(t)\| < \varepsilon/2$ for all $t \geq \tau_{a_2}$. So, finally we achieve that

$$\|e(t)\| \leq \|e_1(t)\| + \|e_2(t)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $t \geq \tau_a = \max\{\tau_{a_1}, \tau_{a_2}\}$.

To reconstruct the fault, we apply similar arguments as it is done in Edwards et al. (2000). Note that in the event-triggering case $\dot{e}_2(t) \neq 0$ for almost all $t \in [t_k, t_{k+1})$ since sliding motion does not take place on the manifold $\{e \in \mathbb{R}^n : e_2 = 0\}$. However, by observing $\|e_2(t)\| \leq \varepsilon/2$ for all $t \geq \tau_a$, one can find a constant $L > 0$ such that $\|\dot{e}_2(t)\| \leq L$ for all $t \geq \tau_a$. So, there exists a constant $\varepsilon_a > 0$ (depending on $\varepsilon$) such that $\|\widetilde{Q}^{-1}_{22}\nu_2(t) - f_a(t)\| < \varepsilon_a$ for all $t \geq \tau_a$. This completes the proof. □

## 5. SIMULATION RESULTS

We consider a linearized model of a passenger vehicle's rigid body dynamics (Edwards et al. (2003)) to demonstrate the proposed methodology for estimating the actuator fault. The model is described by

$$\dot{x} = \begin{bmatrix} -3.9354 & 0 & 0 & -14.7110 \\ 0 & 0 & 0 & 1 \\ 1 & 14.9206 & 0 & 1.6695 \\ 0.7287 & 0 & 0 & -2.1963 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.8116 \end{bmatrix} (u + f_a)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

where $x = [\, x_1 \; x_2 \; x_3 \; x_4 \,]^\top$ represents the average of yaw rate and lateral velocity, orientation of the vehicle, (lateral) deviation from the desired lane position, and the yaw rate, respectively. Deviation (lateral) from the desired lane position and the yaw rate are measurable quantities, i.e., $y = [\, y_1 \; y_2 \,]^\top = [\, x_3 \; x_4 \,]^\top$ is the output vector.

Observe that the disturbance and fault enter through the input channel, i.e., $Q = B$. We consider the fault signal as

$$f_a(t) = \begin{cases} 0 & \text{if } 0 \leq t < 3.5, \\ 5 & \text{if } 3.5 \leq t \leq 5, \\ 0 & \text{otherwise.} \end{cases}$$

Since the system is already in canonical form, we observe that the pair $(\overline{\mathbf{A}}_{11}, \overline{\mathbf{A}}_{2111})$ is observable, where

$$\overline{\mathbf{A}}_{11} = \begin{bmatrix} -3.9354 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{A}}_{2111} = [\, 1 \;\; 14.9206 \,].$$

Now, transforming the system into the new coordinate, $z$, given by

$$z = \begin{bmatrix} 1 & 0 & 1.6078 & 0 \\ 0 & 1 & -0.7153 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x,$$

we obtain $\widetilde{B} = B$, $\widetilde{C} = C$ and the matrix $\widetilde{A}_{11} = \begin{bmatrix} -2.3276 & 23.9887 \\ -0.7153 & -10.6724 \end{bmatrix}$ which is Hurwitz. Then, for $N_1 = 10I_2$, we compute $P_1$ using (11) as $P_1 = \begin{bmatrix} 1.4323 & 2.3292 \\ 2.3292 & 5.7040 \end{bmatrix}$.

Similarly, considering $\widetilde{A}^s_{22} = \text{diag}\{-5, -5\}$, we obtain $P_2 = \text{diag}\{1, 1\}$ for $N_2 = 10I_2$ (from (7)). Here, the switching gain, $M = 18$ is chosen for designing the robust observer. To implement the discontinuous terms, we approximate them by some continuous functions. In this case, we use the following approximation:

$$\text{sign}(e_{2_i}) \approx \frac{e_{2_i}}{|e_{2_i}| + \gamma_i}$$

for some $\gamma_i > 0$, $i = 1, 2$. Here, $\gamma_1 = \gamma_2 = 0.075$. To reconstruct the fault signals, we pass the (approximated) discontinuous signal through a low-pass filter, $\tau_f \dot{x}_f = -x_f + \nu_2$, where $\tau_f > 0$ is the filter coefficient so that its output $x_f$ gives an estimate of fault signal. We take $\tau_f = 0.002$. Now, pick $c_1 = 10$ and $c_2 = 5$. Then, we choose $\varepsilon = 0.2734$ and $\alpha = 0.0031$ so that they satisfy (13) and $\alpha < \varepsilon/2$. Let $\sigma = 0.96$. Let us design a feedback control law. The sliding surface, $Ez = 0$, is chosen for the plant where $E = [\, 0.1209 \; 6.5475 \; 4.9657 \; 1 \,]$ such that the eigenvalues of the reduced order dynamics are placed at $-4$, $-3.5$ and $-2$. Then, we present the control law as

$$u = -(E\widetilde{B})^{-1}(E\widetilde{A}\hat{z} + K_1\text{sign}(E\hat{z}))$$

where $K_1 = 19$ is the controller switching gain. Given all these design parameters, we simulate the system for 10 secs with the initial conditions as $\hat{z}(0) = [\, 0 \; 0 \; 0 \; 0 \,]^\top$ and $z(0) = [\, 0.5 \; -1 \; 0.1 \; 0.3 \,]^\top$. Clearly, $z(0), \hat{z}(0) \in \Pi$. In Fig. 1, it is seen that the estimation error is bounded by the desired bound within 1.2 secs, which implies that the state trajectories remain bounded within $\Pi$ for all time. In Fig. 2, we plot the reconstructed fault signal, which is computed using the switching function $\nu_2$. It is observed that the SMO can estimate the fault under the event-triggered feedback with acceptable accuracy. The absence of the Zeno phenomenon in event-based communication

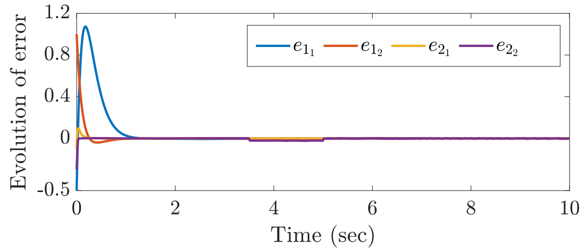


Fig. 1. Convergence of state estimation errors.

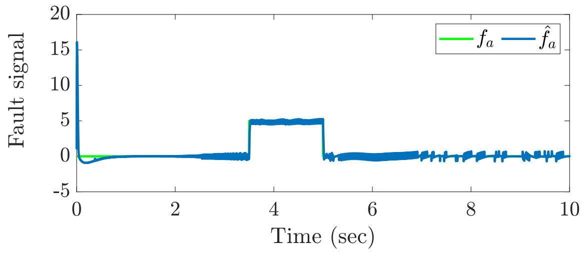


Fig. 2. Event based estimated fault ($\hat{f}_a$) versus the actual fault ($f_a$).

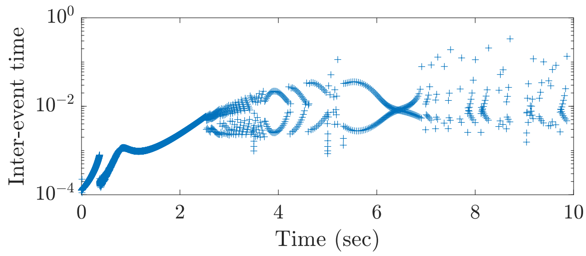


Fig. 3. Variation of inter-event time with time.

is observed in Fig. 3, which ensures the feasibility of the proposed strategy.

## 6. CONCLUSION

We have shown that using an event-based data transmission strategy, the estimated states always stay in the close vicinity of the actual states despite the presence of external fault signals. As the estimation error is bounded practically, it paves the way to reconstruct the fault affecting the system. The precision of fault estimation can be modified by reducing the state estimation error bound, which depends on the triggering threshold. However, it increases the number of triggering instants with a decrease in the triggering parameter. The proposed strategy can be applied to the system operating under network constraints.

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