Robust state estimation via twisting observer ⋆

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Abstract

In this paper, we propose the twisting observer for robustly estimating the states of a second-order uncertain system. It is well-known that the twisting algorithm achieves the second-order sliding mode condition using the sign information of both the plant states. When all the states are not available, the twisting algorithm cannot be extended to the estimation problem directly. Hence, our proposed observer approximates this unknown sign term for the non-measurable state in the twisting observer with a delayed output-based switching function. It is shown that under some suitable gain conditions, the observer can guarantee any arbitrary accuracy of estimation error for some appropriate delay parameter which can be varied online. The advantage of this observer is that one needs to control only the delay parameter to achieve the desired steady-state accuracy, unlike the additional nonlinear functions found in the existing observers. The application of this observer to output feedback stabilization is also discussed. The separation principle for the closed-loop system is shown by designing the gains of controller and observer gains independently. Finally, we illustrate the design and performance of the proposed state observation through a numerical example.

Key words: Robust state estimation, sliding mode observer, twisting observer.

1 Introduction

One of the inherent problems of sliding mode observers is the compromised steady-state accuracy of the estimation process in computer simulation. This is attributed mainly to the high-gain nature of switching terms and non-Lipschitz functions that cause the estimation error to increase abruptly in a sampling interval ([6]). Although these switching terms often create difficulties in numerical implementation, they are essential to guarantee robustness (i.e., insensitive) against uncertainties for robust state estimation. This is the reason why the realization of sliding mode observers/differentiators has gained a lot of attention in the literature, e.g., see [7, 12] and references cited therein. It is shown that for small sampling periods, the real-time implementation of these observers can achieve better accuracy despite the influence of external perturbations ([14]). Indeed, the recent advancement in digital technology has made it possible to process the information at a faster rate, i.e., with smaller sampling periods. Under this assumption, the discretization of higher-order sliding mode observers has been studied extensively in the past decade ([1, 3]). The non-Lipschitz functions in these observers further deteriorate the estimation accuracy ([3]). To address these challenges, an observer may be constructed by eliminating these terms so that their effect on the steady-state value may be neglected. Hence, the goal of this paper is to construct a sliding mode observer containing only switching terms.

The well-known twisting algorithm has only two switching terms depending on the plant states ([12]). Extension of this structure to the estimators demands the information plant states; thus, it limits the applicability of the twisting algorithm to the observation directly. Fortunately, replacing the discontinuous terms with some approximation through *regularization* allows the state estimation even in the presence of disturbance, albeit subject to some accuracy [13]. This modification comes at a price of practical estimation compared to the finite (or asymptotic) convergence of estimated states. Nevertheless, this is an important outcome because the practical accuracy is comparable with that of well-known observers. Based on this observation, our paper makes an attempt to synthesize the twisting observer and shows the robust estimation of states for an uncertain plant in the practical sense.

In this paper, we propose a twisting observer using only output information for a second-order uncertain plant. Since the plant state is not known, we use a delayed approximation

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to realize the unknown switching function in the twisting observer. It is shown that the delay time can be chosen to bound the estimation error within any desired bound. In fact, it is possible to vary the delay unit online without affecting the convergence of the observer dynamics. It may be noted that one may also replace this delayed approximation with its equivalent sampled version of the error function in the numerical implementation. In this case, the accuracy of the estimation is dependent on the sampling period. We also illustrate a general application of this observer to output feedback stabilization. Here, an output feedback-based twisting controller is designed to stabilize the plant under uncertainty. It is shown that the controller and observer gains can be designed independently, establishing the separation principle for the closed-loop system. As a result, the plant and observer trajectories are bounded within any given bound with the output-based twisting controller.

The paper is organized as follows: Section 2 describes the problem statement. The twisting observer is proposed in Section 3. The convergence analysis of estimation error is also discussed here. The application of the proposed observer to the output feedback stabilization problem is given in Section 4. A numerical demonstration of the proposed algorithm is presented in Section 5. Finally, the concluding remarks and future scopes are given in Section 6.

2 Problem statement

We consider a perturbed second-order plant

$$\dot{x}_1 = x_2 \tag{1a}$$
$$\dot{x}_2 = u + d \tag{1b}$$

with its output

$$y = x_1 \tag{1c}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^2$ is the plant state vector, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is the input disturbance to the system.

Assumption 1 *The disturbance d is bounded by some known bound $d_0 > 0$, i.e., $|d(t)| \leqslant d_0$ for all $t \geqslant 0$.*

Our goal is to design a second-order sliding mode observer for the system (1). The observer is to be synthesized based on the twisting algorithm using only output information. However, since the twisting algorithm requires the (sign of) output and its derivative, the observer cannot be realized as a direct extension of its controller counterpart. Here, we explore a twisting-like observer such that its states converge to the plant state subject to some accuracy. To be more specific, we address the following problem in our paper.

Problem Given the system (1), design a twisting-based second-order sliding mode observer

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \kappa(z_1, y, u) \end{aligned} \tag{2}$$

where $\kappa : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the output error-based switching function, which ensures that for every $\varepsilon > 0$, there is a $T \geqslant 0$ such that the observer state $z(t) = \mathrm{col}(z_1(t), z_2(t))$ satisfies $\|x(t) - z(t)\| \leqslant \varepsilon$ for all $t \geqslant T$.

The above problem deals with the practical estimation of the plant states ([10]), and it can have numerous applications in the areas, such as fault detection and isolation ([9]), cyber security ([11]), etc. Indeed, discontinuous observers have benefits over their counterparts in these applications because they can identify the anomaly behavior of the system by treating them as uncertainties.

3 Robust state estimation

This section describes the main result of the paper. We present a second-order sliding mode observer based on the twisting algorithm for estimating the states. The convergence of the estimation error is also discussed.

3.1 Twisting observer

The twisting algorithm requires the sign information of both the state variables to guarantee the convergence of system trajectories to the origin in a finite time. In the observer case, it is not possible to know the exact sign of a state variable, which is not the output of the system. So, we modify the twisting algorithm to synthesize an observer so that the state estimation can be made comparable with their counterparts.

Denote by $e_i = x_i - z_i$, for $i = 1, 2$ the estimation error of the observation process. Then, we propose the following twisting observer

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= K_1 \mathrm{sign}(e_1(t)) + K_2 \mathrm{sign}(e_1(t) - e_1(t-\tau)) + u(t) \end{aligned} \tag{3}$$

where $K_1 > 0$ and $K_2 > 0$ are observer gains, $\tau > 0$ is the time delay and $e_1(\theta) = 0$ for $\theta \in [-\tau, 0)$. Here, the design parameter τ decides the (steady-state) accuracy of the estimation process, which will be illustrated later. It may be observed that the proposed observer (3) does not rely on the other state except for the plant output (1). Hence, the proposed structure can be implemented using plant output and input directly.

The dynamics (3) uses the output information on the interval $[t - \tau, t]$ along with their instantaneous values. This is required to compensate for the derivative of output error in the twisting algorithm. It can be argued that the delayed output function can approximate the sign of error derivative subject to some accuracy, and this finally decides the quality of estimation. We show in our analysis that by adjusting the parameter τ, possibly online, the estimation accuracy can be controlled as per the design specification.

Remark 1 *The delayed function in* (3) *can be realized easily by a discrete approximation when the output measurements are obtained from a sampler. This can be shown by setting* $t - t_i = \tau$ *for any* $t \in [t_i, t_{i+1})$ *and any integer* $i \geqslant 0$ *that* $e_1(t) - e_1(t-\tau) = e_1(t) - e_1(t_i)$. *Then, all the arguments of the delayed function also apply to the sampled version of the approximation. Yet, both representations carry different meanings in the implementation, and we did not explore those aspects here except for their equivalence.*

Remark 2 *The delayed term in the observer* (3) *may also be replaced with* $\mathrm{sign}((e_1(t) - e_1(t-\tau))/\tau)$ *as another approximation for* e_2. *The latter term ensures the exact sign of* e_2 *when* τ *approaches* 0, *which is however not possible to obtain in the delayed function of* (3). *Since both realizations are equivalent when the delay is positive, we use the simple delayed version of the switching term for the sake of convenience.*

3.2 Convergence analysis

We state now the main result of this paper.

Theorem 1 *Consider the system* (1) *and the observer* (3). *If the gains satisfy*

$$K_1 > K_2 + d_0 \quad \textit{and} \quad K_2 > d_0, \tag{4}$$

then for every $\varepsilon > 0$, *there exist* $\tau > 0$ *and* $T \geqslant 0$ *such that* $\|x(t) - z(t)\| \leqslant \varepsilon$ *for all* $t \geqslant T$.

PROOF. Using (1) and (3), the dynamics of the estimation process can be written as

$$\begin{aligned}\dot{e}_1(t) &= e_2(t)\\ \dot{e}_2(t) &= -K_1 \mathrm{sign}(e_1(t)) - K_2 \mathrm{sign}(e_1(t) - e_1(t-\tau))\\ &\quad + d(t).\end{aligned} \tag{5}$$

The above dynamics differ from that of the standard twisting structure with respect to the sign of the second state variable. Yet, the modified term in (5) is powerful enough to show comparable performance with the exact twisting structure. The boundedness of estimation error can be established by applying the arguments of twisting controller subject to the selection of the suitable design parameter τ. For any given $\varepsilon > 0$, let us define the sets $\Omega_i(\varepsilon) := \{e = \mathrm{col}(e_1, e_2) \in \mathbb{R}^2 : |e_i| \leqslant \varepsilon\}$ where $i = 1, 2$, and $\Omega(\varepsilon) := \{e \in \mathbb{R}^2 : \|e\| \leqslant \varepsilon\}$.

The right-hand side of (5) is discontinuous on the (Lebesgue) measure zero set $\{e \in \mathbb{R}^2 : e_1 = 0\}$; thus, it can be understood as delay differential inclusion for the points on this set. Hence, we interpret (5) by the inclusion

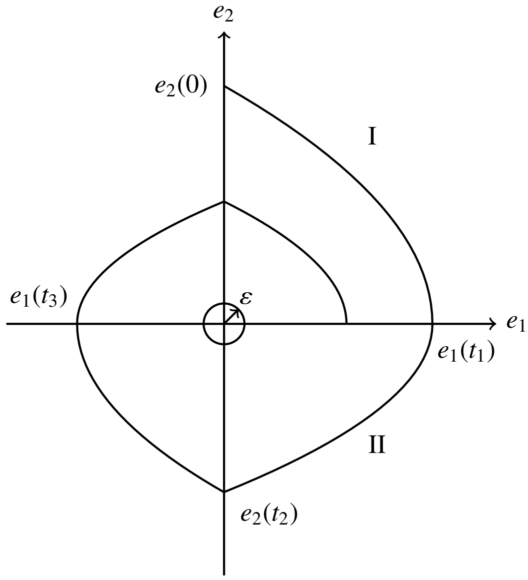


Fig. 1. Error trajectory of the twisting observer.

$$\dot{e}(t) \in \mathcal{F}(t, e(t), e(t-\tau))$$

on this discontinuous set, where $\mathcal{F} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^2 \times C_2 \rightrightarrows \mathbb{R}^2$ is the set-valued functional and C_2 is a set of continuous functions defined on $\mathbb{R}^2$ ([2]). It can be seen that the above inclusion satisfies the basic conditions for the existence of a solution ([2, Theorem 2.2]). Then, by (absolute) continuity of the solution to (5) at a time instant η, we understand that there exists a $\delta > 0$ such that $|t-\eta| \leqslant \delta$ implies $\|e(t)-e(\eta)\| \leqslant \varepsilon$. Hence, it follows that $|e_1(t)-e_1(\eta)| \leqslant \varepsilon$ for all $|t-\eta| \leqslant \delta$. By taking $\eta = t-\tau$, we can write $|e_1(t)-e_1(t-\tau)| \leqslant \varepsilon$ whenever $|\tau| \leqslant \delta$. Now, notice from (5) that e_1 increases (decreases) monotonically if and only if $e_2 > 0$ ($e_2 < 0$). This leads to the following: if $|\tau| = |t-\eta| \leqslant \delta$ such that $e_2(\eta) > \varepsilon$, then $0 < e_1(t)-e_1(t-\tau) \leqslant \varepsilon$. Similarly, $-\varepsilon \leqslant e_1(t)-e_1(t-\tau) < 0$ whenever $|\tau| = |t-\eta| \leqslant \delta$ with $e_2(\eta) < -\varepsilon$. From this, we conclude that $\mathrm{sign}(e_1(t) - e_1(t-\tau)) = \mathrm{sign}(e_2(t))$ whenever $e(t)$ is outside the set $\Omega_2(\varepsilon)$ for any $t \geqslant 0$.

To begin with the convergence analysis, we first consider the case that the trajectory of the error dynamics (5) starts from anywhere in $\mathbb{R}^2$ but outside the set $\Omega_2(\varepsilon)$, i.e., $|e_2(0)| > \varepsilon$. Without loss of generality, assume that $e_2(0) > \varepsilon$, Clearly, one has $\mathrm{sign}(e_1(t)-e_1(t-\tau)) = \mathrm{sign}(e_2(t))$ whenever $e_2(t) > \varepsilon$ for $t \geqslant 0$. Then, it can be observed that the error trajectories of (5) in the first quadrant remain bounded by the trajectories of the dynamics

$$\dot{e}_1 = e_2 \quad \text{and} \quad \dot{e}_2 = -(K_1 + K_2 - d_0).$$

On solving this with the initial point $(0, e_2(0))$, the equation of motion can be given by

$$e_2^2(t) = e_2^2(0) - 2(K_1 + K_2 - d_0)e_1(t) \tag{6}$$

for all $0 \leqslant t \leqslant t_1$, where t_1 is the time instant at which the curve hits the axis $e_2 = 0$, i.e., $e_2(t_1) = 0$. This is depicted by segment I in Fig. 1.

Note that $\mathrm{sign}(e_1(t)-e_1(t-\tau))$ may not be equal to $\mathrm{sign}(e_2(t))$ whenever $-\varepsilon < e_2(t) < 0$ for some $t \geqslant 0$ because the delayed function in (5) cannot change the sign abruptly after zero crossing. Despite this, it can be seen from (5) that the trajectory enters the fourth quadrant after crossing the line $e_2 = 0$ due to $K_1 > K_2 + d_0$ (cf. (4)). The trajectories in this region can be bounded by segment II, whose dynamics is governed by the following equations

$$\dot{e}_1 = e_2 \quad \text{and} \quad \dot{e}_2 = -(K_1 + K_2 + d_0).$$

If t_1' denotes the time at which $e_2(t_1') = -\varepsilon$, then the equation of segment II for the interval $t_1 < t \leqslant t_1'$ is found to be

$$e_2^2(t) = 2(K_1 + K_2 + d_0)(e_1(t_1) - e_1(t)).$$

As soon as the trajectory crosses the line $e_2 = -\varepsilon$, it holds that $\mathrm{sign}(e_1(t) - e_1(t-\tau)) = \mathrm{sign}(e_2(t))$ for any $e_2(t) < -\varepsilon$ and $t \geqslant 0$. Here, the error dynamics (5) satisfy the inequality $-(K_1 - K_2 + d_0) \leqslant \dot{e}_2 \leqslant -(K_1 - K_2 - d_0)$. The gain condition $K_1 > K_2 + d_0$ ensures that the error trajectory always converges towards the e_2 axis in the fourth quadrant. Then, the dynamics of segment II can be given by

$$\dot{e}_1 = e_2 \quad \text{and} \quad \dot{e}_2 = -(K_1 - K_2 + d_0)$$

whose solutions bound every other possible trajectory of (5) can be given by

$$e_2^2(t) = e_2^2(t_1') + 2(K_1 - K_2 + d_0)(e_1(t_1') - e_1(t)) \tag{7}$$

for all $t_1' < t \leqslant t_2$. Here, t_2 is the time instant at which the trajectory hits the line $e_1 = 0$, i.e., $e_1(t_2) = 0$. Now, with some simple calculations, it can be shown that

$$\begin{aligned} e_2^2(t_2) &= \frac{(K_1 - K_2 + d_0)}{(K_1 + K_2 - d_0)} e_2^2(0) + \left(1 - \frac{K_1 - K_2 + d_0}{K_1 + K_2 + d_0}\right)\varepsilon^2 \\ &= \frac{(K_1 - K_2 + d_0)}{(K_1 + K_2 - d_0)} e_2^2(0) + \frac{2K_2}{(K_1 + K_2 + d_0)}\varepsilon^2 \end{aligned}$$

which eventually leads to

$$\frac{e_2^2(t_2)}{e_2^2(0)} = \frac{(K_1 - K_2 + d_0)}{(K_1 + K_2 - d_0)} + \frac{2K_2}{(K_1 + K_2 + d_0)} \frac{\varepsilon^2}{e_2^2(0)}.$$

Note that $(K_1 - K_2 + d_0)/(K_1 + K_2 - d_0) < 1$ directly follows from (4). Then, without loss of generality, we can choose a sufficiently small $\varepsilon > 0$ so that the right side term in the above becomes less than unity. Therefore, the successive intercepts on the e_2 axis decrease whenever $|e_2(0)| > \varepsilon$. By repeating this argument sequentially, we can conclude that $|e_2(t_{2i})| \leqslant \varepsilon$ for some integer $i \geqslant 0$. Then, by (uniform) continuity of the error trajectory, it follows that $\|e(t)\| \leqslant \varepsilon$ for all $t \geqslant t_{2i}$. Hence, the estimation error remains bounded within the set $\Omega(\varepsilon)$ in a finite time T $(= t_{2i})$. This completes the proof. □

Remark 3 *The error dynamics* (5) *satisfies the homogeneity property as that of a standard twisting algorithm. Indeed, it may be verified easily that the error dynamics is homogeneous of degree* -1 *with the dilation* $d_\lambda : (e_1, e_2) \mapsto (\lambda^2 e_1, \lambda e_2)$ *for any* $\lambda > 0$. *This property can be used to show the finite time stability of the origin of error dynamics. However, this is not recalled in our proof because the origin is not asymptotically stable.*

3.3 Estimation under noisy measurements

Let us consider the system (1a)-(1b) with the output

$$y_m = x_1 + v \tag{8}$$

where v is the measurement noise. Suppose that the noise v is bounded by some bound v_0, i.e., $|v(t)| \leqslant v_0$ for all $t \geqslant 0$. This is a common assumption often used in the sliding mode literature to analyze the effect of noise on the system performance [6, 8].

The noise consists of high-frequency signals, which are amplified by the switching components due to their high-gain nature. However, the observer can show robustness against the measurement noise of small magnitudes. We discuss here the boundedness of the estimation error in the presence of this class of noises. With the noisy measurement, the observer (3) yields the error dynamics as

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) \\ \dot{e}_2(t) &= -K_1 \mathrm{sign}(e_1(t) + v(t)) + d(t) \\ &\quad - K_2 \mathrm{sign}(e_1(t) - e_1(t-\tau) + v(t) - v(t-\tau)). \end{aligned} \tag{9}$$

The convergence of the above dynamics can be analyzed in a similar manner as it is done in Theorem 1. For the case that $|e_1(t)| > v_0$ and $|e_1(t) - e_1(t-\tau)| > 2v_0$ for any $t \geqslant 0$, the error dynamics become the same as that of (5). Moreover, when $|e_1| \leqslant v_0$, the trajectory leaves the axis $e_1 = 0$ due to either $e_2 > 0$ and $e_2 < 0$. As a result, the convergence of error trajectory to the neighborhood of $e = 0$ can be shown using similar arguments as in Theorem 1. In this case, the steady-state bound is also dependent on the noise. Thus, the robustness to measurement noise is guaranteed for the proposed observer.

The filtering is another approach adopted to smoothen the plant output corrupted by the noise. In sliding mode-based differentiators, the filtering is found to be very effective in suppressing a large class (high-amplitude) of noises ([8]).

4 Output feedback stabilization

This part of the paper illustrates an application of the twisting observer to the output feedback stabilization problem. We consider here the twisting controller for stabilizing the plant robustly. The well-known state feedback twisting controller is given by

$$u = -L_1 \text{sign}(x_1) - L_2 \text{sign}(x_2) \tag{10}$$

where L_1 and L_2 are some positive gains ([12]). It is shown in [12] that proper selection of these gains guarantees the finite-time stabilization of the origin. However, due to the practical estimation of states by the observer (3), the output feedback twisting controller

$$u = -L_1 \text{sign}(x_1) - L_2 \text{sign}(z_2) \tag{11}$$

can only achieve practical stabilization. Note that here we use the output function directly in the controller instead of the estimator output. The closed-loop dynamics can be given by the plant dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -L_1 \text{sign}(x_1) - L_2 \text{sign}(x_2 - e_2) + d \end{aligned} \tag{12}$$

and the observer dynamics (3). The stability analysis for the closed-loop system can be presented by establishing the separation principle of controller and observer structure.

Remark 4 *The output-based twisting controller can also be realized directly as* $u(t) = -L_1 \text{sign}(x_1(t)) - L_2 \text{sign}(x_1(t) - x_1(t-\tau))$ *for some delay* $\tau > 0$. *By repeating similar arguments, like in Theorem 1, it is not hard to show that the states can be bounded within any bound. In fact, the similar fact is highlighted in [4, 5] in the real-time implementation. In our work, we use an observer-based twisting controller to demonstrate the output feedback stabilization. The importance of these observer-based controllers can be found in recent applications such as cyber-security, where observers are employed to detect cyber-attacks ([11]).*

Theorem 2 *Consider the closed-loop plant* (12) *and the observer dynamics* (3). *Suppose that* $x(0), z(0) \in \mathbb{R}^2$. *If*

$$L_1 > L_2 + d_0 \quad \text{and} \quad L_2 > d_0 \tag{13}$$

then for every $\varepsilon_c > 0$, *the plant trajectories remain bounded for all time, and there exists a time* $T_c \geqslant 0$, *and the observer delay* $\tau > 0$ *such that* $\|x(t)\| \leqslant \varepsilon_c$ *for all* $t \geqslant T_c$.

PROOF. Let $\varepsilon_c > 0$ be any given scalar. Now, suppose that the system (1) is driven by the state feedback control (10). It is well known that the system trajectories remain bounded within an invariant region on the phase plane. We choose a (shaded) positively invariant region Λ such that it is contained within the set $\{x \in \mathbb{R}^2 : \|x\| \leqslant \varepsilon_c\}$ as shown in Fig. 2. Moreover, since the closed-loop system (12) has a continuous solution, there exists a $\tau_c > 0$ such that the continuity at the point $x(t) = 0$ for some time $t \geqslant 0$ implies $\|x(t) - x(t - \tau_c)\| \leqslant \varepsilon_c$ when $|\tau_c| \leqslant \delta_c$ for some $\delta_c > 0$. Let

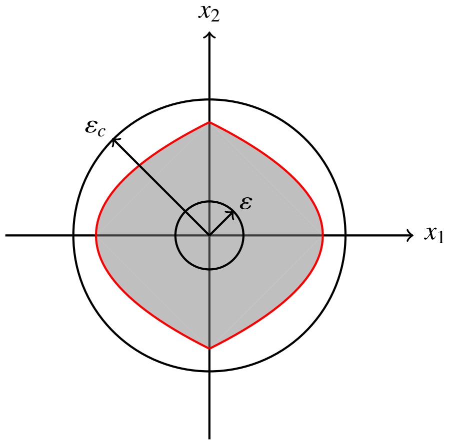


Fig. 2. Positively invariant region of the twisting controller.

$\varepsilon < \varepsilon_c$ be a positive scalar such that $\{x \in \mathbb{R}^2 : \|x\| \leqslant \varepsilon\} \subset \Lambda$. From Theorem 1, we pick a $\tau \in (0, \tau_c)$ such that the estimation error $e(t)$ is bounded by ε on the interval $[t - \tau, t]$, i.e., $\|e(t) - e(t - \tau)\| \leqslant \varepsilon$.

Suppose that $|e_2(0)| > \varepsilon$. Then, by Theorem 1, there exists a time $T \geqslant 0$ such that $\|e(t)\| \leqslant \varepsilon$ for all $t \geqslant T$. Let us analyze the behavior of plant dynamics over the intervals $[0, T)$ and $[T, \infty)$ separately.

Consider the set $\{x \in \mathbb{R}^2 : |x_2| > |e_2|\}$. When the plant trajectories are outside this set, one can verify that $\text{sign}(x_2 - e_2) = \text{sign}(x_2)$. Hence, the trajectories of (12) will converge to the origin for $|x_2| > |e_2|$. On the other hand, for $|x_2| \leqslant |e_2|$, the sign of the second term in the controller (11) cannot be identified correctly. In this case, one may write the system dynamics (12) as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -L_1 \text{sign}(x_1) + L_2 \text{sign}(x_2) + d. \end{aligned} \tag{14}$$

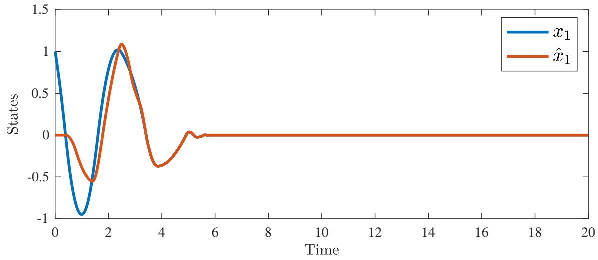
Applying the standard arguments to the above systems, one can show

$$x_2^2(t_2) = \frac{(L_1 + L_2 + d_0)}{(L_1 - L_2 - d_0)} x_2^2(0)$$

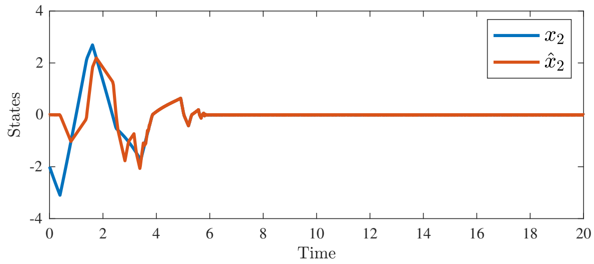
which guarantees the boundedness of plant trajectories over any finite time interval. Thus, the plant trajectories are bounded on the interval $[0, T)$.

Since $|e_2(t)| \leqslant \varepsilon$ for all $t \geqslant T$, the identity $\text{sign}(x_2(t) - e_2(t)) = \text{sign}(x_2(t))$ holds for any $|x_2(t)| > \varepsilon$ and any $t \geqslant T$. Thus, the trajectory $x(t)$ converges to the set Λ for $t \geqslant T$, and it eventually remains bounded within this set in a finite time T_c. Hence, $x(t) \in \{x \in \mathbb{R}^2 : \|x\| \leqslant \varepsilon_c\}$ for all $t \geqslant T_c$. This completes the proof. □

Remark 5 *Theorem 2 shows the practical stabilization for the system* (1) *with output feedback controller* (11). *Since*



(a) State trajectories with time.



(b) State trajectories with time.

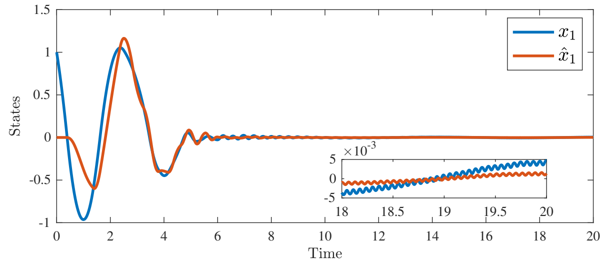
Fig. 3. The trajectories of the closed-loop system without measurement noise.

the observer guarantees the estimation of states practically, the plant trajectories converge to the equilibrium point in the practical sense. Moreover, the separation principle also applies because the controller and observer gains are designed independently, and the trajectories converge to the respective desired point eventually.

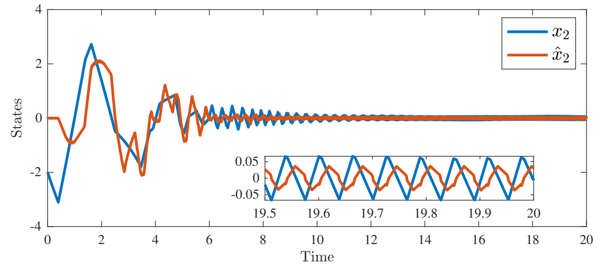
5 Simulation results

In this section, we demonstrate the estimation process using the proposed twisting observer. Consider the system (1) with the disturbance $d = \sin t$. Let the plant be stabilized by a state feedback-based twisting controller (10) with $L_1 = 3$ and $L_2 = 1.5$ since $d_0 = 1$. These gain conditions ensure the finite time stability of $x = 0$ for the closed-loop system with the feedback law (10) ([12]). Since the state vector is not available in our case, we synthesize the twisting observer (3) to estimate the states.

We use the Euler discretization with a step size of $h = 0.0001$ to simulate the system. This step size is also used to approximate the delayed term in the observer (3) with its sampled version, i.e., $\tau = h$. Observer gains are chosen as $K_1 = 5$ and $K_2 = 2$ as per Theorem 1. The initial values of the closed-loop plant and observer are taken as $x = [\,2 \;\; 8\,]^\top$ and $z = [\,0 \;\; 0\,]^\top$, respectively. The closed loop system with state feedback control and observer is simulated in MATLAB, and the final results are shown in Fig. 3. It is seen that observer states converge to the plant state subject to the given accuracy. This steady-state accuracy can be improved by choosing the appropriate value of the design parameter



(a) State trajectories with time.



(b) State trajectories with time.

Fig. 4. The trajectories of the closed-loop system with measurement noise.

τ, possibly varying online. The plant states also converge to the vicinity of the origin.

The performance of the observer with the noise signal $0.01\sin(100t)$ is also studied. Fig. 4 shows the plot of plant and observer trajectories with the considered noisy signal. The observer states converge to the plant states, but it remain bounded in steady-state subject to the noise bound. This shows the robustness of the proposed observer against the measurement noise.

6 Conclusion

A robust observer based on the twisting algorithm was proposed to estimate the states of a second-order uncertain plant. As the twisting algorithm uses two sign terms for both the state components, an observer cannot be designed by extending the twisting structure directly to the estimation case. We modified the twisting algorithm by approximating the unknown sign term with an output-based delay function through regularization. By choosing this delay, any desired estimation accuracy can be obtained using this observer. An application of this observer to the output feedback stabilization was also presented. The separation principle was established for the proposed observer, which guarantees the convergence of plant and observer states to any given bound.

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